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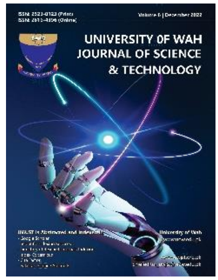

A New K-Product Generalized Transformation: Investigating Weibull Distribution as the Baseline

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Abstract

This article is about defining and studying an improved technique of parameter induction to a continuous probability distribution through a new G-class of probability models. In particular, the Weibull distribution is used in the defined technique and it is named as KP-W distribution. The importance of this generalization of Weibull distribution comes from its ability to model various kinds of hazard functions such as ascending, descending, first decreasing and then increasing, or constant hazard rate functions. Different properties of this generalized modified model have been deliberated along with raw moments and functions which can generate moments, quantiles, hazard function, Rényi entropy, stress-strength parameter, order statistics, the average time to wait until served and average remaining life. Maximum likelihood (ML) estimation of the proposed G class and its sub-model, the KP-W is also presented. Finally, the KP-W model is judged for its goodness to fit using data sets from different fields to showcase its practical applications.

Keywords

G-class of probability distributions, Weibull distribution, hazard function, entropy measures and maximum likelihood estimation.

1. Introduction

In reliability theory, the Weibull distribution is a well-known and widely used probability model. It has been commonly used in analyzing lifetime datasets but it does not provide a better fit on lifetime datasets in certain situations. To overcome these types of weaknesses, many authors have developed different extensions to the Weibull distribution. Recently, [1] used the α -power transformation on the well-known Weibull distribution and obtained the alpha-power Weibull distribution. They applied their model to real-life datasets to show how it works in practice.

Several such modifications are available in the literature which can be used to have a new distribution function (DF) using a baseline DF. Some of the important transformations include

(i) Exponentiated-G class of distributions, defined as $G_1(y;\tau)=[F(y;\tau)]^{\alpha}, \alpha>0$ and presented by [2].

(ii) QRTM is defined as $G_2(y;\tau)=(1+\lambda)F(y;\tau)+\lambda[F(y;\tau)]^2, |\lambda|\le 1$ presented by [3]

(iii) DUS transformation is defined as $G_3(y;\tau)=\dfrac{e^{F(y;\tau)}-1}{e-1}$ and this transformation was proposed by [4].

(iv) A transformation by[5] is defined as $G_4(y;\tau)=\text{Sin}\left(\dfrac{\pi}{2}F(y;\tau)\right)$.

(v) A transformation by [6] is defined as $G_5(y;\tau)=e^{1-\frac{1}{F(y;\tau)}}$.

(vi) A transformation by [7] is defined as $G_6(y;\tau)=1-\dfrac{\log_e[2-F(y;\tau)]}{\log_e[2]}$.

(vii) GDUS transformation proposed by [8] with DF given by $G_7(y;\tau)=\dfrac{e^{\alpha[F(y;\tau)]}-1}{e-1}, \alpha>0$

.

(viii) A transformation using a trigonometric function, suggested by [9] and defined as $G_8(y;\tau)=\text{Sin}\left[\dfrac{\pi}{4}F(y;\tau)(F(y;\tau)+1)\right], \forall x\in R$.

(ix) A transformation initiated by [10] defined as $G_9(y;\tau)=\dfrac{2[F(y;\tau)]}{[1+F(y;\tau)]}$.

(x) The Marshal and Olkin Transformation by [11] has the DF $G_{10}(y;\tau)=1-\dfrac{\alpha(1-F(y;\tau))}{1-(1-\alpha)(1-F(y;\tau))}$.

(xi) Another transformation using the trigonometric function defined as $G_{11}(y;\tau) = \text{Tan}\left(\frac{\pi}{4} F(y;\tau)\right)$ and proposed by [12].

(xii) A generalized family of distributions by [13] having DF as

$$G_{12}(y;\tau) = \frac{e}{e-1}\left[1 - e^{-\left\{1-e^{\left(-\alpha \frac{G(y;\tau)}{\overline{G}(y;\tau)}\right)^{\beta}}\right\}}\right].$$

Similarly, some of the recent modifications or transformations may be found in [14], [15],[16], [17] and [18] , among others. The key contribution of the current paper is based on discussing a generalization of a class of models with DF $G_9(y;\tau)$. This generalization of $G_9(y;\tau)$ is named as KP-Generalized (KP-G) class of distributions. We now present the KP-G class in the next section.

2. KP-G class of distributions

Letting $F(y;\tau)$ as the *DF* of a random quantity *Y* with parameters τ , *DF* of the random variable *Y* following KP-G class of models defined as

$$G_{13}(y;\tau,k) = \frac{kF(y;\tau)}{k-1+F(y;\tau)}. \tag{1}$$

The above defined $G_{13}(y;\tau,k)$ is a complete *DF* for *k* > 1. For $k = 2$, it becomes $G_9(y;\tau)$.

Also, the corresponding probability distribution function (*PDF*) can be derived as

$$g_{13}(y;\tau,k) = \frac{k(k-1)f(y;\tau)}{\{k-1+F(y;\tau)\}^2}. \tag{2}$$

2.1 Linear representation of *DF* and *PDF*

From (2), the $g_{11}(y;\tau,k)$ may be rewritten as

$$g_{13}(y;\tau,k) = k(k-1)f(y;\tau,k)\{k-1+F(y;\tau)\}^{-2} = \frac{kf(y;\tau)}{k-1}\left\{1+\frac{F(y;\tau)}{k-1}\right\}^{-2} \tag{3}$$

Using the binomial expansion $(1+y)^{-2} = \sum_{m=0}^{\infty}\binom{-2}{m} y^{m} = \sum_{m=0}^{\infty}(m+1)(-y)^{m}$ with $|y|<1$, in (3), we get

$$g_{13}(y;\tau,k) = \sum_{m=0}^{\infty}\frac{k(-1)^{m}(m+1)F(y;\tau)^{m} f(y;\tau)}{(k-1)^{m+1}} = \sum_{m=0}^{\infty}\frac{k(-1)^{m} h_{(m+1)}(y;\tau)}{(k-1)^{m+1}},$$

$$g_{13}(y;\tau,k) = \sum_{m=0}^{\infty} w_m h_{(m+1)}(y;\tau). \tag{4}$$

In (4), we have $w_m = \frac{k(-1)^{m}}{(k-1)^{m}}$ and $h_{(m+1)}(y;\tau) = (m+1)F(y;\tau)^{m} f(y;\tau)$ is the *PDF* of the exponentiated class of distributions based on $F(y;\tau)$ with exponentiation parameter $(m+1)$.

Now , using (4), the *CDF* $G_{13}(y;\tau,k)$ may be written as

$$G_{13}(y;\tau,k) = \int_{-\infty}^{y} g_{13}(y;\tau,k)dx = \int_{-\infty}^{x}\sum_{m=0}^{\infty}\frac{(-1)^{m}(j+1)kF(y;\tau)^{m} f(y;\tau)}{(k-1)^{m+1}}dy$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^{m}\int_{-\infty}^{x} kh_{(m+1)}(y;\tau)dy}{(k-1)^{m+1}}$$

$$G_{13}(y;\tau,k) = k\sum_{j=0}^{\infty}\frac{(-1)^{m} H_{(m+1)}(y;\tau)}{(k-1)^{m+1}} = \sum_{j=0}^{\infty} w_m H_{(m+1)}(y;\tau), \tag{5}$$

where $H_{(m+1)}(y;\tau) = F(y;\tau)^{(m+1)}$ is the *DF* of exponentiated class of distributions.

2.2 r^th^ moment and moment generating function(*mgf*)

By definition,

$$\mu_r' = \int_{-\infty}^{\infty} y^r g_{13}\left(y;\tau,k\right)dy = \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} y^r w_m h_{(m+1)}\left(y;\tau\right)dy$$

$$\mu_r' = \sum_{m=0}^{\infty} w_m \int_{-\infty}^{\infty} y^r h_{(m+1)}\left(y;\tau\right)dy = \sum_{m=0}^{\infty} w_m \mu_{r,(m+1)}'$$

In the above expression $\mu_{r,(m+1)}'$ symbolizes the rth raw moment of an exponentiated form of the baseline class of models.

Similarly, *mgf* of KP-G class of models is given by

$$M_Y\left(t\right) = \int_{-\infty}^{\infty} e^{ty} g_{13}\left(y;\tau,k\right)dy = \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} e^{ty} w_m h_{(m+1)}\left(y;\tau\right)dy$$

$$M_Y\left(t\right) == \sum_{j=0}^{\infty} w_m \int_{-\infty}^{\infty} e^{ty} h_{(m+1)}\left(y;\tau\right)dy = \sum_{m=0}^{\infty} w_m M_{Y,(m+1)}\left(t\right),$$

where $M_{Y,(m+1)}\left(t\right)$ is *mgf* associated with an exponentiated form of family of the baseline distributions.

2.3 Survival, hazard and quantile functions

Survival, hazard and quantile functions of the KP-G family of models are respectively given by

$$S\left(y;\tau,k\right) = 1 - G_{13}\left(y;\tau,k\right) = 1 - \frac{kF\left(y;\tau\right)}{k-1+F\left(y;\tau\right)} = \frac{\left(k-1\right)F\left(y;\tau\right)}{k-\left(1-F\left(y;\tau\right)\right)}, \tag{6}$$

$$h\left(y;\tau,k\right) = \frac{g_{13}\left(y;\tau,k\right)}{S\left(y;\tau,k\right)} = \frac{kf\left(y;\tau\right)}{F\left(y;\tau\right)\left(k-1+F\left(y;\tau\right)\right)} = \frac{kr'\left(y;\tau\right)}{\left(k-S'\left(y;\tau\right)\right)}, \tag{7}$$

$$q_u\left(y\right) = F^{-1}\left(\frac{u\left(k-1\right)}{k-u}\right), \tag{8}$$

where $r'\left(y;\tau\right)$ and $S'\left(y;\tau\right)$ are the reverse hazard rate function and survival function of the baseline model $f\left(y;\tau\right)$ and $u \sim Uniform\left(0,1\right)$.

3. KP- Weibull (KP-W) distribution: a special case

By substituting *PDF* and *DF* of the Weibull model as $F\left(y;\tau\right)$ and $f\left(y;\tau\right)$, respectively, in the above expressions in equations (2) and (1), we can obtain the *DF* and *PDF* of KP-W distribution. An important feature of this model is that the newly inducted parameter k can produce certain attractive properties and can fit certain lifetime data-sets better than the previously available generalizations of the Weibull distribution.

As we know that the *PDF and DF* of the Weibull probability model having λ and β as parameters, are expressed as:

$$f\left(y;\lambda,\beta\right)=\lambda\beta y^{\beta-1}e^{-\lambda y^{\beta}}, \tag{9}$$

$$F\left(y;\lambda,\beta\right)=1-e^{-\lambda y^{\beta}}, \tag{10}$$

Using (1), (2), (9) and (10), *DF* and *PDF* of the KP-W model with three parameters k, λ and β are given below.

$$G\left(y;\lambda,\beta,k\right)=\frac{k\left(1-e^{-\lambda y^{\beta}}\right)}{k-e^{-\lambda y^{\beta}}},\ k\geq 1 \text{ and } ,\lambda,\beta \text{ and } x>0 \tag{11}$$

Also the *PDF* against *DF* in (11) can be derived as given below

$$g\left(y;\lambda,\beta,k\right)=k\left(\beta y^{\beta-1}\lambda e^{-\lambda y^{\beta}}\right)\left(k-1\right)\left(k-e^{-\lambda y^{\beta}}\right)^{-2}. \tag{12}$$

We can write , and by using binomial series expansion, we can write, $k^{-2}\left(\left(1-\dfrac{e^{-\lambda y^{\beta}}}{k}\right)^{-2}\right)=\dfrac{1}{k^{2}}\sum_{m=0}^{\infty}\left(m+1\right)\left(\dfrac{e^{-\lambda y^{\beta}}}{k}\right)^{m}$. Consequently, *PDF* of KP-W model is given as

$$g\left(y;\lambda,\beta,k\right)=\left(k-1\right)\left(\lambda\beta y^{\beta-1}\right)\sum_{j=0}^{\infty}\left(j+1\right)\left(\frac{e^{-\lambda y^{\beta}}}{k}\right)^{j+1} \tag{13}$$

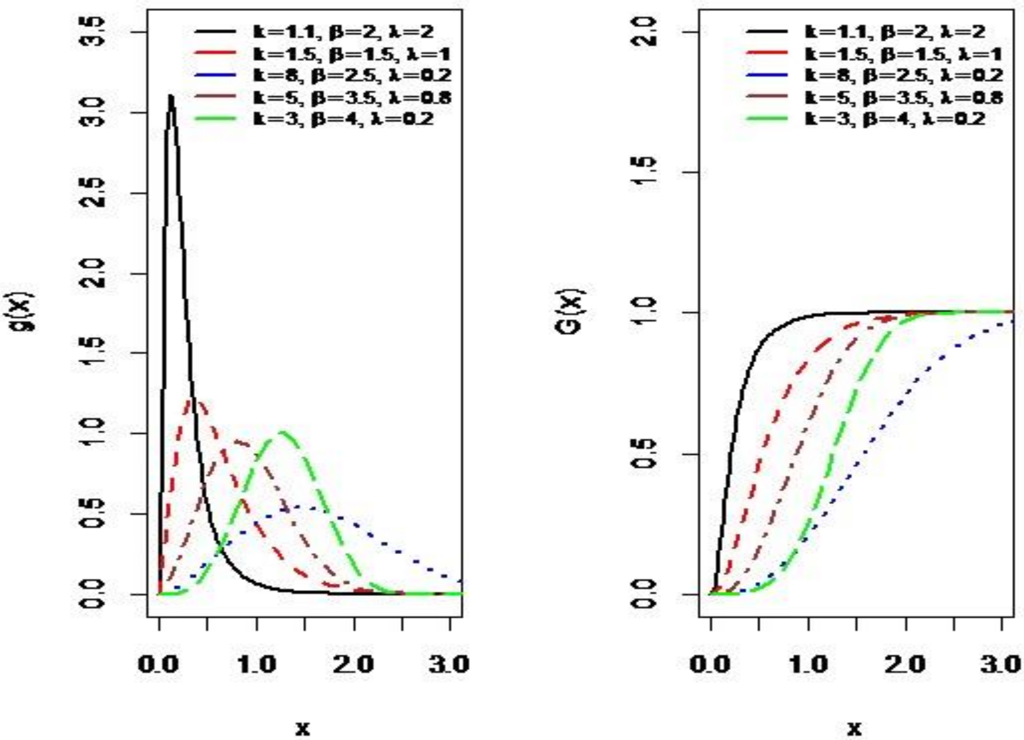

Fig 1: *Graphs of different shapes of the PDF and DF of the KP-W distribution for different parametric values.*

In figure 1 shapes of *PDF* and *DF* of KP-W model at different combinations of parametric values are shown. From these shapes we see that the distribution is highly and positively skewed for smaller values of *k*. If we take large values of parameter *k*, this degree of skewness decreases.

Given below are the certain cases which the KP-W distribution can generalize.

1. If $\lambda = 1$ then $g(y)$ defined in (13) reduces to KP- one parameter Weibull distribution.

2. If $\beta = 1$ then the above $g(x)$ defined in (13) reduces to KP exponential model.
3. If $\beta = 2$, the above $g(y)$ defined in (13) reduces to KP Rayleigh distribution.

3.1 Structural Properties of KP-W Model

Now, we study different characteristics like, hazard function, moments, mean residual life, stress-strength parameter, survival function moment generating function (MGF), Rényi entropy, quantile function and ordered random variables of the KP-W model.

3.1.1 Survival Function

Survival function, $S(y;\lambda,\beta,k)$, of the KP-W model is expressed as:

$$S(y;\lambda,\beta,k) = 1 - G(y;\lambda,\beta,k) = \frac{(k-1)e^{-\lambda y^{\beta}}}{k - e^{-\lambda y^{\beta}}}. \qquad (14)$$

3.1.2 Hazard Function

The hazard or failure rate function of the KP-W distribution is derived as follows.

$$h(y;\lambda,\beta,k) = \frac{g(y;\lambda,\beta,k)}{S(y;\lambda,\beta,k)}$$

$$h(y;k,\lambda,\beta) = \frac{k\lambda\beta y^{\beta-1}}{k - e^{-\lambda y^{\beta}}}. \qquad (15)$$

The Figure 2 showcases the pictorial representation of hazard function of KP-W model. These graphs show that it can be used to handle multiple hazard types.

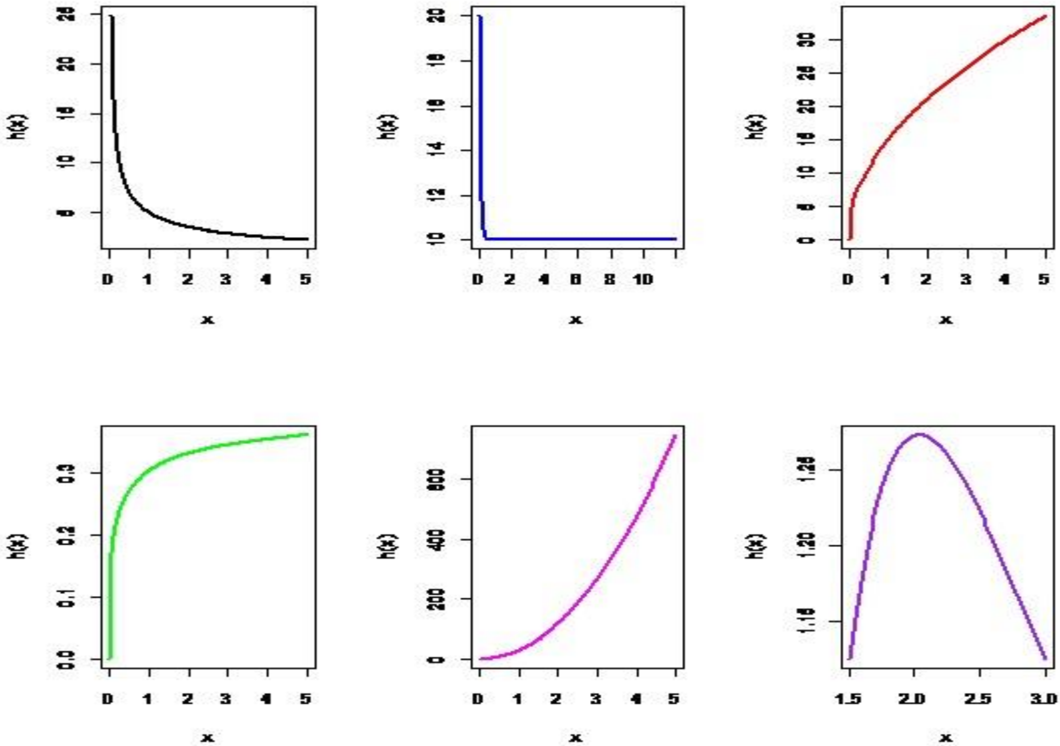

Fig. 2: *Shapes of hazard rates of KG-W distribution against different combinations of parametric values.*

3.1.3 Quantile Function

By equating $G(y;\lambda,\beta,k)=u$, where $u \sim Uniform(0,1)$. On simplifying this expression, the quantile function of KP-W model ca be shown as presented below.

$$Y=\left[-\frac{1}{\lambda}\ln\left\{\frac{k(1-u)}{k-u}\right\}\right]^{\frac{1}{\beta}}. \tag{16}$$

The q^{th} quantile of KP-W model is expressed as:

$$Y_{(q)} = \left[-\frac{1}{\lambda} \ln \left\{ \frac{k(1-q)}{k-q} \right\} \right]^{\frac{1}{\beta}}. \tag{17}$$

Putting $q = 0.5$, median of the KP-W model is given by:

$$Y_{(0.5)} = \left[-\frac{1}{\lambda} \ln \left\{ \frac{k}{2k-1} \right\} \right]^{\frac{1}{\beta}}. \tag{18}$$

3.1.4 Moments

The r^{th} raw moment of the KP-W probability model is expressed as:

$$E\left(Y^r\right) = \int_0^\infty y^r g\left(x; k, \lambda, \beta\right) dy$$

$$= \lambda\beta\left(k-1\right)\int_0^\infty y^r y^{\beta-1} \sum_{m=0}^{\infty}\left(m+1\right)\left(\frac{e^{-\lambda y^\beta}}{k}\right)^{m+1} dy$$

Let $\lambda y^\beta = z$ then $\lambda\beta y^{\beta-1} dy = dz$ and $y^r = \left(\frac{z}{\lambda}\right)^{\frac{r}{\beta}}$. So we have

$$E\left(y^r\right) = \left(k-1\right)\int_0^\infty \left(\frac{z}{\lambda}\right)^{\frac{r}{\beta}} \sum_{m=0}^{\infty}\left(m+1\right)\left(\frac{e^{-z}}{k}\right)^{m+1} dz$$

$$= \frac{\left(k-1\right)}{\lambda^{\frac{r}{\beta}}} \sum_{m=0}^{\infty} \frac{\left(m+1\right)}{k^{m+1}} \int_0^\infty \left(z\right)^{\frac{r}{\beta}+1-1} \left(e\right)^{-z(m+1)} dz$$

$$= \frac{\left(k-1\right)}{\lambda^{\frac{r}{\beta}}} \sum_{m=0}^{\infty} \frac{\left(m+1\right)}{k^{m+1}} \int_0^\infty \left(z\right)^{\frac{r}{\beta}+1-1} \left(e\right)^{-z(m+1)} dz$$

$$=\frac{(k-1)}{\lambda^{\frac{r}{\beta}}}\sum_{m=0}^{\infty}\frac{(m+1)}{k^{m+1}}\frac{\Gamma\left(\frac{r}{\beta}+1\right)}{(m+1)^{\frac{r}{\beta}+1}}$$

$$=\frac{(k-1)}{\lambda^{\frac{r}{\beta}}}\sum_{m=0}^{\infty}\frac{\Gamma\left(1+\frac{r}{\beta}\right)}{k^{m+1}(m+1)^{\frac{r}{\beta}}}. \tag{19}$$

To get mean, put $r = 1$ in (19) and have

$$E(Y)=\frac{(k-1)}{\lambda^{\frac{1}{\beta}}}\sum_{m=0}^{\infty}\frac{\Gamma\left(1+\frac{1}{\beta}\right)}{k^{m+1}(m+1)^{\frac{1}{\beta}}}. \tag{20}$$

Put $r = 2$ in (19) to get second raw moment as:

$$E(Y^2)=\frac{(k-1)}{\lambda^{\frac{2}{\beta}}}\sum_{m=0}^{\infty}\frac{\Gamma\left(1+\frac{2}{\beta}\right)}{k^{m+1}(m+1)^{\frac{2}{\beta}}}. \tag{21}$$

Then, the variance of the KP-W distribution is obtained as:

$$Var(Y)=\frac{(k-1)}{\lambda^{\frac{2}{\beta}}}\sum_{m=0}^{\infty}\frac{\Gamma\left(1+\frac{2}{\beta}\right)}{k^{m+1}(m+1)^{\frac{2}{\beta}}}-\left[\frac{(k-1)}{\lambda^{\frac{1}{\beta}}}\sum_{m=0}^{\infty}\frac{\Gamma\left(1+\frac{1}{\beta}\right)}{k^{m+1}(m+1)^{\frac{1}{\beta}}}\right]^2. \tag{22}$$

3.1.5 MGF of KP-W model

For a random variable Y having KP-W model with *PDF* $g(y;k,\lambda,\beta)$, the *MGF* is derived

As below.

$$M_Y(t)=E\left(e^{tY}\right)=\int_0^{\infty}e^{ty}g(y;k,\lambda,\beta)\,dy$$

By Maclaurin's series expansion we can write $e^{ty} = \sum_{r=0}^{\infty} \frac{(ty)^r}{r!}$.

Using the above expansion and *PDF of* the KG-W distribution we have,

$$M_Y(t) = E\left(e^{ty}\right) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{(ty)^r}{r!} (k-1)\lambda\beta y^{\beta-1} \sum_{m=0}^{\infty} (m+1) \left(\frac{e^{-\lambda y^{\beta}}}{k} \right)^{m+1} dy$$

Using substitution $\lambda y^{\beta} = z$ and after further simplification, we can get the following expression,

$$M_Y(t) = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(m+1)(k-1)}{k^{m+1} \lambda^{\frac{r}{\beta}}} \frac{t^r}{r!} \int_0^{\infty} z^{\frac{r}{\beta}+1-1} e^{-z(m+1)} dz$$

$$M_Y(t) = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(m+1)(k-1)}{k^{m+1} \lambda^{\frac{r}{\beta}}} \frac{t^r}{r!} \frac{\Gamma\left(1+\frac{r}{\beta}\right)}{(m+1)^{1+\frac{r}{\beta}}}. \tag{23}$$

3.1.6 MRL and MWT

For a random variable Y having $S(y;\tau)$ as its survival function, the MRL function is represented as estimated residual lifetime after a specific time point *s*, that is,

$$\mu(s) = E(X - s | X > s)$$

$$\mu(s) = \frac{1}{S(s)} \left(E(s) - \int_0^{s} y g(y;\tau) dy \right) - s, \tag{24}$$

where

$$\int_0^{s} y g(y;\tau) dy = \int_0^{s} y\lambda(k-1)\beta y^{\beta-1} \sum_{m=0}^{\infty} (m+1) \left(\frac{e^{-\lambda y^{\beta}}}{k} \right)^{m+1} dy$$

$$= \frac{(k-1)}{\lambda^{\frac{1}{\beta}}} \sum_{m=0}^{\infty} \left(\frac{1}{k^{m+1} (m+1)^{\frac{1}{\beta}}} \right) \Gamma\left(\frac{1}{\beta} + 1; (m+1)\lambda s^{\beta} \right), \tag{25}$$

and $\Gamma(\alpha;y)=\int_0^y e^{-y}y^{\alpha-1}dy$ is an incomplete gamma function.

By substituting equations (14),(20) and (25) in (24), we can write $\mu(t)$ as:

$$\mu(s)=\frac{k-e^{-\lambda s^{\beta}}}{e^{-\lambda s^{\beta}}(k-1)}\frac{(k-1)}{\lambda^{\frac{1}{\beta}}}\left(\sum_{m=0}^{\infty}\frac{\Gamma\left(1+\frac{1}{\beta}\right)}{k^{m+1}(m+1)^{\frac{1}{\beta}}}-\sum_{m=0}^{\infty}\frac{\Gamma\left(1+\frac{1}{\beta};(m+1)\lambda s^{\beta}\right)}{k^{m+1}(m+1)^{\frac{1}{\beta}}}\right)-s$$

$$\mu(s)=\frac{k-e^{-\lambda s^{\beta}}}{e^{-\lambda s^{\beta}}\lambda^{\frac{1}{\beta}}}\sum_{m=0}^{\infty}\frac{1}{k^{m+1}(m+1)^{\frac{1}{\beta}}}\left[\Gamma\left(\frac{1}{\beta}+1\right)-\Gamma\left(\frac{1}{\beta}+1;(m+1)\lambda s^{\beta}\right)\right]-s\,. \tag{26}$$

The MWT tells about the time taken by an object before its failure, under the assumption that the failure has occurred within the time interval starting from 0 to s. The expression for MWT is calculated as:

$$\overline{\mu}(s)=s-\frac{1}{G(s)}\int_0^s xg(x;\lambda,\beta,k)\,dx\,. \tag{27}$$

Substituting equations (11) and (25) in (27) we have,

$$\overline{\mu}(s)=s-\frac{\left(k-e^{-\lambda s^{\beta}}\right)(k-1)}{\left(1-e^{-\lambda s^{\beta}}\right)k\lambda^{\frac{1}{\beta}}}\sum_{m=0}^{\infty}\frac{\Gamma\left(1+\frac{1}{\beta}1;(m+1)\lambda s^{\beta}\right)}{k^{j+1}(m+1)^{\frac{1}{\beta}}}\,. \tag{28}$$

3.1.7 Stress-Strength(S-S) Parameter

Let Y_1 and Y_2 are two independent continuous random variables, where $Y_1 \sim$ KP-W(k_1,λ_1,β) and $Y_2 \sim$ KP-W(k_2,λ_2,β). The S-S constant, say R, is given by:

$$R=\int_{-\infty}^{\infty} g_1(y;\lambda_1,\beta,k_1)G_2(y;\lambda_2,\beta,k_2)\,dy\,. \tag{29}$$

By substituting (11) and (12) in (29), we have,

$$R=\int_0^{\infty}\frac{k_1(k_1-1)\lambda_1\beta y^{\beta-1}e^{-\lambda_1 y^{\beta}}}{\left(k_1-e^{-\lambda_1 y^{\beta}}\right)^2}\frac{k_2\left(1-e^{-\lambda_2 y^{\beta}}\right)}{k_2-e^{-\lambda_2 y^{\beta}}}dy$$

$$R=\frac{(k_1-1)\lambda_1}{k_1}\int_0^\infty \beta y^{\beta-1}e^{-\lambda_1 y^\beta}\left(1-e^{-\lambda_2 y^\beta}\right)\left(1-\frac{e^{-\lambda_1 y^\beta}}{k_1}\right)^{-2}\left(1-\frac{e^{-\lambda_2 y^\beta}}{k_2}\right)^{-1}dy. \qquad (30)$$

Using binomial series, we get,

$$\left(1-\frac{e^{-\lambda_1 y^\beta}}{k_1}\right)^{-2}=\sum_{i=0}^\infty (1+i)\left(\frac{e^{-\lambda_1 y^\beta}}{k_1}\right)^i$$

$$\left(1-\frac{e^{-\lambda_2 y^\beta}}{k_2}\right)^{-1}=\sum_{l=0}^\infty \left(\frac{e^{-\lambda_2 y^\beta}}{k_2}\right)^l.$$

So, R is written as:

$$R=\frac{(k_1-1)\lambda_1}{k_1}\sum_{i=0}^\infty\sum_{l=0}^\infty\frac{(1+i)}{k_1^i k_2^l}\left[\int_0^\infty \beta y^{\beta-1}e^{-x^\beta(\lambda_1(1+i)+\lambda_2 l)}dy-\int_0^\infty \beta y^{\beta-1}e^{-x^\beta(\lambda_1(1+i)+\lambda_2(1+l))}dy\right]$$

By making substitution $y^\beta = z$, R reduces to,

$$R=\frac{(k_1-1)\lambda_1}{k_1}\sum_{i=0}^\infty\sum_{l=0}^\infty\frac{(1+i)}{k_1^i k_2^l}\left[\frac{1}{\lambda_1(1+i)+\lambda_2 l}-\frac{1}{\lambda_1(1+i)+\lambda_2(1+l)}\right]. \qquad (31)$$

3.1.8 Rényi Entropy

By entropy we can measure the difference in uncertainty of a random variable *Y*. The expression for Rényi Entropy is:

$$R_\varphi(y;\lambda,\beta,k)=\frac{1}{1-\varphi}\log\left[\int_{-\infty}^\infty \{g(y;\lambda,\beta,k)\}^\varphi dy\right],\ \phi>0,\phi\neq 1.$$

Using here, the *PDF* of the KG-W distribution, discussed earlier, we have,

$$R_\varphi(y;\lambda,\beta,k)=\frac{1}{1-\varphi}\log\left[\int_0^\infty\left\{\frac{(k-1)\lambda k\beta y^{\beta-1}e^{-\lambda y^\beta}}{\left(k-e^{-\lambda y^\beta}\right)^2}\right\}^\varphi dy\right]$$

$$R_\varphi(y;\lambda,\beta,k)=\frac{\varphi}{1-\varphi}\log k(k-1)$$

$$+\frac{1}{1-\varphi}\log\left[\int_0^\infty (\lambda\beta)^\varphi\ y^{\varphi(\beta-1)}e^{-\phi\lambda y^\beta}k^{-2\varphi}\left(1-\frac{e^{-\lambda y^\beta}}{k}\right)^{-2\varphi}dy\right]$$

Now, applying the binomial series expansion formula, we can write,

$$\left(1-\frac{e^{-\lambda y^\beta}}{k}\right)^{-1}=\sum_{i=0}^{\infty}\binom{2\varphi+i-1}{2\varphi-1}\left(\frac{e^{-\lambda y^\beta}}{k}\right)^i$$

.Now, substituting $\lambda y^\beta(i+\varphi)=z$ and after simplification the expression for Rényi Entropy becomes,

$$R_\varphi(y;\lambda,\beta,k)=\frac{\varphi}{1-\varphi}\log k(k-1)$$

$$+\frac{1}{1-\varphi}\log\left[\sum_{i=0}^{\infty}\binom{2\varphi+i-1}{2\varphi-1}\frac{\lambda^{\frac{1}{\beta}(\varphi-1)}\beta^{\varphi-1}}{k^{2\varphi+i}(i+\varphi)^{\varphi-\frac{1}{\beta}(\varphi-1)}}\int_0^\infty e^{-z}z^{\varphi-\frac{1}{\beta}(\varphi-1)-1}dy\right]$$

$$R_\varphi(y;\lambda,\beta,k)=\frac{\varphi}{1-\varphi}\log k(k-1)$$

$$+\frac{1}{1-\varphi}\log\left[\sum_{i=0}^{\infty}\binom{2\varphi+i-1}{2\varphi-1}\frac{\lambda^{\frac{1}{\beta}(\varphi-1)}\beta^{\varphi-1}}{k^{2\varphi+i}(i+\varphi)^{\varphi-\frac{1}{\beta}(\varphi-1)}}\Gamma\left(\varphi-\frac{1}{\beta}(\varphi-1)\right)\right].\qquad(32)$$

3.1.9 Ordered random variables

Assuming $y_1, y_2, ...y_n$ as a randomly drawn *n* sample values from the KP-W probability model, and $Y_{j:n}$ represents the j^{th} ordered statistic, the *PDF* of $Y_{j:n}$ is defined below.

$$g_{j:n}(y;\lambda,\beta,k)=\frac{n!}{(n-j)!(j-1)!}g(y;\lambda,\beta,k)\left[1-G(y;\lambda,\beta,k)\right]^{n-j}\left[G(y;\lambda,\beta,k)\right]^{j-1}\qquad(33)$$

Substituting (11) and (12) in (33) we get

$$g_{j:n}(y;\lambda,\beta,k)=\frac{n!}{(n-j)!(j-1)!}\frac{k(k-1)\lambda\beta y^{\beta-1}e^{-\lambda y^{\beta}}}{\left(k-e^{-\lambda y^{\beta}}\right)}\left[1-\frac{k\left(1-e^{-\lambda y^{\beta}}\right)}{k-e^{-\lambda y^{\beta}}}\right]^{n-j}\left[\frac{k\left(1-e^{-\lambda y^{\beta}}\right)}{k-e^{-\lambda y^{\beta}}}\right]^{j-1}. \quad (34)$$

3.1.10 Estimation of Parameters

By using the different estimation approaches, we find estimates of the three constants k,λ and β of KP-W probability distribution.

(i) Maximum Likelihood Estimation

Suppose that a random sample $Y_1, Y_2, \ldots Y_n$ of size n from KP-W distribution is randomly drawn, then the log-likelihood function is given by

$$l=n\ln\{k(k-1)\lambda\beta\}-\lambda\sum_{i=1}^{n}y_i^{\beta}+\sum_{i=1}^{n}\ln\left[\frac{1}{\left(k-e^{-\lambda y_i^{\beta}}\right)^2}\right]+(\beta-1)\sum_{i=1}^{n}\ln y_i .$$

To get the MLEs of k,λ and β, we differentiate the above expression with respect to $\lambda\ \beta$ and k and equate to zero. This gives the following system of equations

$$\frac{\partial l}{\partial k}=-2\sum_{i=1}^{n}\ln\left[\frac{1}{\left(k-e^{-\lambda y_i^{\beta}}\right)}\right]+\frac{n(2k-1)}{k(k-1)}=0, \quad (35)$$

$$\frac{\partial l}{\partial \lambda}=\frac{n}{\lambda}-2\sum_{i=1}^{n}\ln\left[\frac{e^{-\lambda x_i^{\beta}}y_i^{\beta}}{\left(k-e^{-\lambda y_i^{\beta}}\right)}\right]-\sum_{i=1}^{n}y_i^{\beta}=0, \quad (36)$$

$$\frac{\partial l}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln y_i-\lambda\sum_{i=1}^{n}y_i^{\beta}\ln y_i-2\sum_{i=1}^{n}\ln\left[\frac{e^{-\lambda y_i^{\beta}}y_i^{\beta}\ln y_i}{\left(k-e^{-\lambda y_i^{\beta}}\right)}\right]=0. \quad (37)$$

We can find the numerical solutions of MLEs of parameters k,λ and β with the help of any software like R, Matlab, or Mathematica.

(ii) Least Squares Estimators

The least squares estimators $\hat{k}_{LSE}$; $\hat{\lambda}_{LSE}$ and $\hat{\beta}_{LSE}$ of the unknown parameters k , λ and β of the KP-W distribution can be obtained by minimizing the following function

$$LS(k,\lambda,\beta)=\sum_{i=1}^{n}\left[G(y_i|k,\lambda,\beta)-\frac{i}{n+1}\right]^2 \tag{38}$$

$$LS(k,\lambda,\beta)=\sum_{i=1}^{n}\left[\frac{k\left(1-e^{-\lambda y_i^{\beta}}\right)}{k-e^{-\lambda y_i^{\beta}}}-\frac{i}{n+1}\right]^2, \tag{39}$$

with respect to k, λ and β. That is least squares estimators can be obtained by solving the following system of differential equations given in (40)-(42).

$$\frac{\partial LS(k,\lambda,\beta)}{\partial k}=-2\sum_{i=1}^{n}A_i^1\left[\frac{k\left(1-e^{-\lambda y_i^{\beta}}\right)}{k-e^{-\lambda y_i^{\beta}}}-\frac{i}{n+1}\right]=0, \tag{40}$$

$$\frac{\partial LS(k,\lambda,\beta)}{\partial \lambda}=2\sum_{i=1}^{n}A_i^2\left[\frac{k\left(1-e^{-\lambda y_i^{\beta}}\right)}{k-e^{-\lambda y_i^{\beta}}}-\frac{i}{n+1}\right]=0, \tag{41}$$

$$\frac{\partial LS(k,\lambda,\beta)}{\partial \beta}=2\sum_{i=1}^{n}A_i^3\left[\frac{k\left(1-e^{-\lambda y_i^{\beta}}\right)}{k-e^{-\lambda y_i^{\beta}}}-\frac{i}{n+1}\right]=0, \tag{42}$$

where $A_i^1=\frac{e^{-\lambda y_i^{\beta}}\left(1-e^{-\lambda y_i^{\beta}}\right)}{\left(k-e^{-\lambda y_i^{\beta}}\right)^2}$, $A_i^2=\frac{k(k-1)y_i^{\beta}e^{-\lambda y_i^{\beta}}}{\left(k-e^{-\lambda y_i^{\beta}}\right)^2}$, and $A_i^3=\frac{k(k-1)\lambda\beta y_i^{\beta-1}e^{-\lambda y_i^{\beta}}}{\left(k-e^{-\lambda y_i^{\beta}}\right)^2}$

(iii) Weighted Least Squares Estimation

The weighted least squares estimators $\hat{k}_{WLSE}$; $\hat{\lambda}_{WLSE}$ and $\hat{\beta}_{WLSE}$ of the unknown parameters k, λ and β of the KP-W distribution can be obtained by minimizing the following function with respect to k, λ and β.

$$WLS(k,\lambda,\beta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n+1-i)}\left[G(y_i|k,\lambda,\beta)-\frac{i}{n+1}\right]^2 \tag{43}$$

$$WLS(k,\lambda,\beta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n+1-i)}\left[\frac{k\left(1-e^{-\lambda y_i^{\beta}}\right)}{k-e^{-\lambda y_i^{\beta}}}-\frac{i}{n+1}\right]^2. \tag{44}$$

That is, the weighted least squares estimators can be obtained by solving the following system of differential equations given in (45)-(47).

$$\frac{\partial WLS(k,\lambda,\beta)}{\partial k} = -2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n+1-i)} A_i^1 \left[\frac{k\left(1-e^{-\lambda y_i^\beta}\right)}{k-e^{-\lambda y_i^\beta}} - \frac{i}{n+1} \right] = 0 \tag{45}$$

$$\frac{\partial LS(k,\lambda,\beta)}{\partial \lambda} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n+1-i)} A_i^2 \left[\frac{k\left(1-e^{-\lambda y_i^\beta}\right)}{k-e^{-\lambda y_i^\beta}} - \frac{i}{n+1} \right] = 0, \tag{46}$$

$$\frac{\partial LS(k,\lambda,\beta)}{\partial \beta} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n+1-i)} A_i^3 \left[\frac{k\left(1-e^{-\lambda y_i^\beta}\right)}{k-e^{-\lambda y_i^\beta}} - \frac{i}{n+1} \right] = 0 \tag{47}$$

where $A_i^t, (t=1,2,3)$ are same as given above.

(iv) Cramer-Von-Mises Estimators

The Cramer-Von-Mises estimators $\hat{k}_{CVME}$; $\hat{\lambda}_{CVME}$ and $\hat{\beta}_{CVME}$ of the unknown parameters k, λ and β of the KP-W distribution can be obtained by minimizing the following function with respect to k, λ and β.

$$CVMS(k,\lambda,\beta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[G(y_i|k,\lambda,\beta) - \frac{2i-1}{2n} \right]^2 \tag{48}$$

$$CVMS(k,\lambda,\beta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[\frac{k\left(1-e^{-\lambda y_i^\beta}\right)}{k-e^{-\lambda y_i^\beta}} - \frac{2i-n}{2n} \right]^2. \tag{49}$$

That is, the Cramer-Von-Mises estimators can be obtained by solving the following system of differential equations given in (50)-(52).

$$\frac{\partial CVMS(k,\lambda,\beta)}{\partial k} = -2\sum_{i=1}^{n} A_i^1 \left[\frac{k\left(1-e^{-\lambda y_i^\beta}\right)}{k-e^{-\lambda y_i^\beta}} - \frac{2i-1}{2n} \right] = 0, \tag{50}$$

$$\frac{\partial CVMS(k,\lambda,\beta)}{\partial \lambda} = 2\sum_{i=1}^{n} A_i^2 \left[\frac{k\left(1-e^{-\lambda y_i^\beta}\right)}{k-e^{-\lambda y_i^\beta}} - \frac{2i-1}{2n} \right] = 0, \tag{51}$$

$$\frac{\partial CVMS(k,\lambda,\beta)}{\partial \beta} = 2\sum_{i=1}^{n} A_i^3 \left[\frac{k\left(1-e^{-\lambda y_i^\beta}\right)}{k-e^{-\lambda y_i^\beta}} - \frac{2i-1}{2n} \right] = 0, \tag{52}$$

where $A_i^t, (t=1,2,3)$ are same as given above.

4. Simulations Study

To analyze the precision of the estimates of parameters of KP-W distribution, a simulations study is conducted. The parameters are estimated by using ML estimation. We generated 1000 sample of sizes n = 20, 50 and 100 from the KP-W distribution. To generate the samples of different sizes from KP-W distribution, we have used the Quantile function of this model defined above. From each sample, we computed the estimates by using the method of ML estimation. The simulated averages and MSEs over 1000 repetitions are given in the following Table 1. From the Table 4.1, it is clear that the simulated values are close enough to the true values of parameters and if we increase the size of sample, the mean squared errors of these estimates decrease.

Table 4.1: Simulated averages and mean squared errors (MSE) of MLEs for the parameters β, λ and k of the KP-W distribution.

n = 20								
Parmaeters			Average MLEs			MSEs		
β	λ	k	$\hat{\beta}$	$\hat{\lambda}$	$\hat{k}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{k}$
0.5	0.1	1.1	0.548	0.198	1.37	2.292	0.342	0.158
		2.0	0.558	0.209	2.786	1.762	0.269	5.872
1.0	1.0	1.1	1.79	1.98	1.65	0.950	0.399	0.491
		2.0	1.85	1.985	2.86	3.048	0.586	0.146
n = 50								
0.5	0.1	1.1	0.527	0.179	1.26	1.056	0.165	0.104
		2.0	0.546	0.186	2.479	0.792	0.119	0.786
0..5	0.1	1.1	1.48	1.75	1.52	0.554	0.236	0.254
		2.0	1.67	1.82	2.47	1.672	0.337	0.104
n =100								
0.5	0.1	1.1	0.512	0.114	1.17	0.612	0.095	0.064
		2.0	0.521	0.134	2.124	0.456	0.068	0.387
0.5	0.1	1.1	1.16	1.35	1.29	0.342	0.141	0.175
		2.0	1.18	1.43	2.34	0.871	0.187	0.069

5. Applications

In this portion, we discuss the applications of KP-W distribution on those data-sets that belong to real life to prove the signification and flexibility of this model.

(i) First real-life application

The first data-set corresponds the monthly taxes revenue in Egypt (measured in 1000 million Egyptian pounds) for the period January 2006 to November 2010. This data-set is taken from [19]. This data-set consists of following observations,

5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6,9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8

We analyzed this data by the proposed technique, the KP-W distribution, and compared with different existing models mentioned in the Table 5.1 below.

Table 5.1: AICs and BICs computed after fitting different distributions on Data set of Tax Revenues.

Model	AIC	BIC
Lgistic Eponential	389.35	393.51
Weibull	398.58	402.73
Exponential	427.01	429.09
Marshal Olikin Logistic Eponential	387.34	393.58
KP-W	**386.79**	**393.02**
Lomax	429.0136	43.17
Exponential Lomax	400.5883	406.82

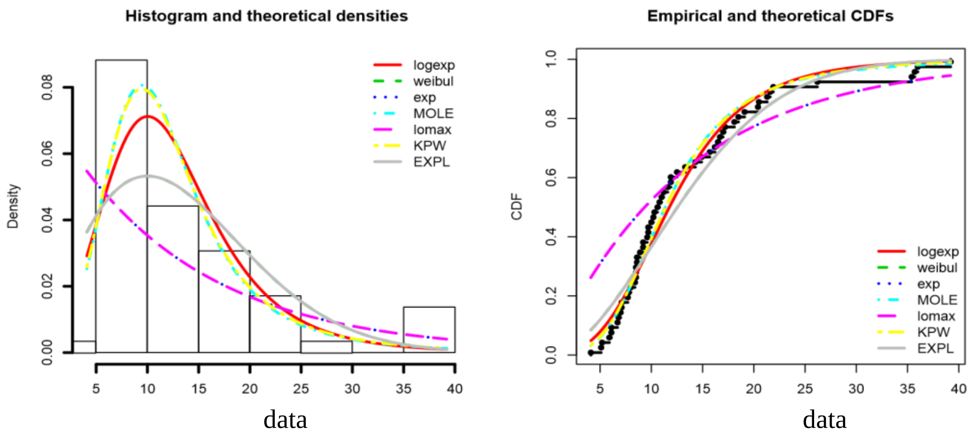

***Fig 3:** Density Plots and CDF plots of different fitted distributions for Tax revenue data-set.*

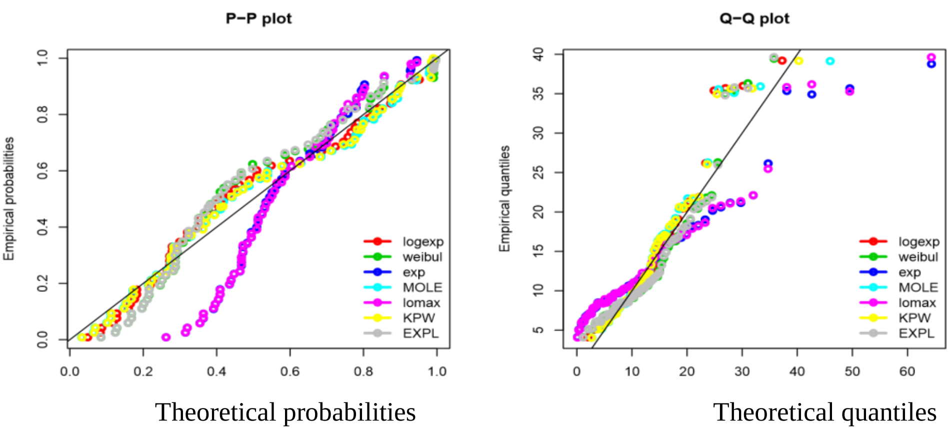

Fig 4: *PP and QQ plots for Tax revenue dataset and fitted distributions.*

(ii) Second real-life application

The values in this set data set are Brake-Pads Life times for 98 Cars (units per 1000 kilometers) taken from [20] The data-set consists of following values:

Model	AIC	BIC
Logitic Exponential	1131.04	1136.21
Weibull	924.24	929.40
Expoential	1021.56	1024.15
Marshal Olik Logistic Exponential	921.46	929.21
KP-W	**921.20**	**928.96**
Kumaraswamy Lomax	923.223	933.5630
Lomax	1023.5669	1028.7368
Expoentiated Lomax	926.2954	934.0503
Weibull Lomax	923.4776	933.8175
Expoentiated Weibull	921.2813	929.0362

38.7, 69.6, 86.7, 79.5, 49.2, 74.8, 43.8, 55.0,42.4, 32.9, 100.6, 46.8, 73.8, 51.5, 67.6, 124.5, 46.7, 31.8, 89.5, 92.5, 44.1, 77.6, 60.3, 110.0, 61.9, 63.7, 103.6, 101.2,39.3, 83.0, 82.6, 59.4, 49.8, 24.8, 88.0, 27.8, 46.3, 68.8, 42.4, 33.6, 56.2, 68.8, 68.9, 69.0, 50.5, 89.1, 95.7, 75.2, 54.9, 65.0, 78.1, 58.4, 54.0, 65.1, 83.6, 105.6, 49.2, 59.3, 18.6, 56.2,44.8, 53.9, 92.6, 55.9, 15.8, 30.9, 57.3, 83.8, 107.8, 47.4, 34.3, 123.5,81.6, 61.4, 105.6, 69.0,45.2, 72.8, 20.8, 101.9, 124.6, 54.0, 52.0, 87.6, 64.0, 37.2, 77.2, 38.8, 83.0, 44.2, 68.9, 74.7, 143.6, 50.8, 78.7,43.4, 65.5, 165.5

The KP-W model is fitted to the above data and its performance is compared with that of the different competitor models mentioned in the Table 5.2 below.

Table 5.2: AICs and BICs computed after fitting different distributions on Data set of Brake pads Lifetimes.

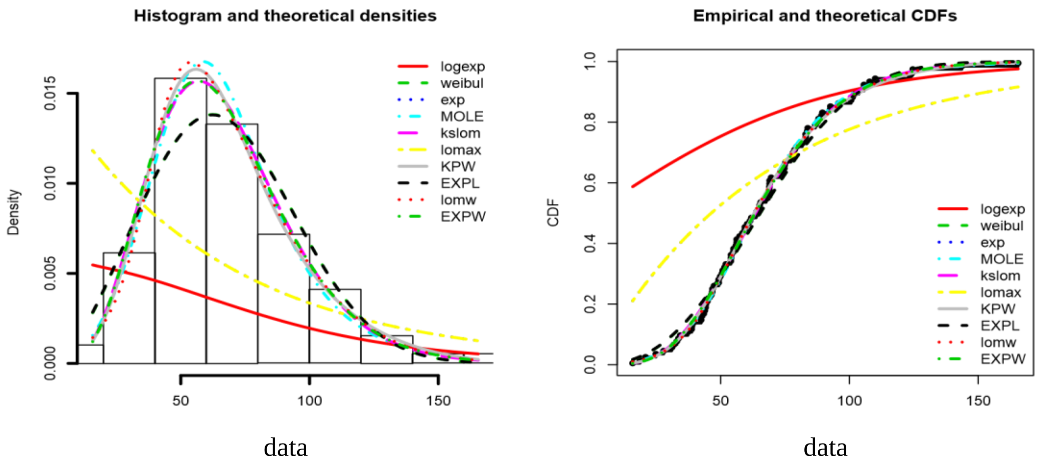

Fig. 5: *Density and CDF plots for Brake pads lifetimes dataset and distributions*.

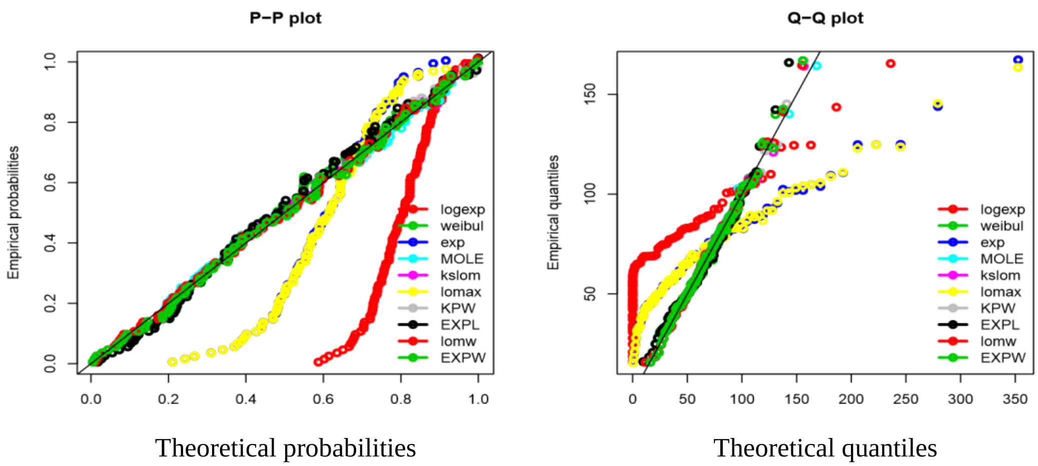

Fig. 6: *PP and QQ plots for Brake pads lifetime dataset and fitted distributions.*

6. Conclusive remarks

A new G class of probability distributions, named as KP-class of distributions, is presented. The suggested KP-class of distributions is based on the induction of an additional parameter resulting in more tractable distribution for analyzing the lifetime data. Further, we have derived expressions for moments, Rényi Entropy, ordered statistics, MRL and stress-strength parameter for the KP-W distribution, a sub-model of the proposed family. The method of ML estimation is used to estimate the parameters. The fitting of this KP- W distribution on two data-sets from real world also illustrated that the suggested distribution provides satisfactory goodness of fit measures in comparison to competitive probability distributions. From the tables 2 and 3, representing AICs and BICs of different models fitted on two different data sets, we can see that the AICs and BICs obtained after fitting the KP-W distribution are minimum and also we can see from the density, Q-Q, *CDF* and P-P plots that the KP-W probability distribution turns out to be a better choice for modeling the two real life data in comparison to the other existing probability distributions.

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References

[1] M. Nassar, M. A. Alzaatreh, , M. Mead, and Abo-Kasem, O. Alpha power Weibull distribution: Properties and applications. Communications in Statistics-Theory and Methods, Vol. 46(20), pp. 10236–10252, 2017.

[2] R. C. Gupta, R. D. Gupta, and P. L. Gupta, P. L. Modeling failure time data by Lehman alternatives', Communications in Statistics Theory and Methods, Vol. 27, pp. 887-904, 1998

[3] W. T. Shaw and I. R. and Buckley. The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv preprint arXiv:0901.0434.2009

[4] D. Kumar, U. Singh and S. K. Singh. A method of proposing new distribution and its application to Bladder cancer patients' data. Journal of Statistics Applications & Probability, Vol. *2*(3), pp. 235-245, 2015.

[5] D. Kumar, U. Singh and S. K. Singh. A new distribution using sine function-its application to bladder cancer patients' data. Journal of Statistics Applications & Probability, Vol. *4*(3), pp. 417-427, 2015.

[6] D. Kumar, U. Singh and U. Singh, U. Life time distributions: Derived from some minimum guarantee distribution. Sohag Journal of Mathematics, Vol.*4*(1), pp. 7-11, 2017.

[7] S. K. Maurya, A. Kaushik, R. K. Singh, S. K. Singh, and U. Singh, U. A new method of proposing distribution and its application to real data. Imperial Journal of Interdisciplinary Research, Vol. 2(6), pp. 1331– 1338, 2016.

[8] S. K. Maurya, A. Kaushik, S. K. Singh, and U. Singh. A new class of distribution having decreasing, increasing, and bathtub-shaped failure rate. Communications in Statistics-Theory and Methods, Vol. 46(20), pp. 10359–10372, 2017

[9] Z. Mahmood and C Chesneau. A new sine-G family of distributions: properties and applications. hal-02079224, 2019.

[10] N. Kyurkchiev. A new transmuted cumulative distribution function based on the Verhulst logistic function with application in population dynamics. Biomath Communications, Vol. 4(1), 2017 DOI: 10.11145/bmc.2017.05.132

[11] A. N. Marshall and I. Olkin. A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families, Biometrika, Vol. 84, pp. 641-552,1997

[12] L. Souza, W. R. Junior, C. C. R. de Brito, C. Chesneau, R. L. Fernandes and T. A. E. Ferreira, Tan-G class of trigonometric distributions and its applications. CUBO, A Mathematical Journal, Vol. 23(1), pp. 01–20, 2021.

[13]I. Elbatal, S. M. Alghamdi, F. Jamal, S. Khan, E. M. Almetwally and M. Elgarhy., Kavya-Manoharan Weibull-G family of distributions: Statistical inference under progressive type-II censoring scheme, Vol. 87(2),pp. 191-223, 2023

[14] A. Shafiq, S. A. Lone, T. N. Sindhu, Y. El-Khatib, Q. M. Al-Mdallal, and T. Muhammad. A new modified Kies Fréchet distribution: Applications of mortality rate of Covid-19, Results in Physics, 28:104638. doi:10.1016/j.rinp.2021.104638, 2021.

[15] T. N. Sindhu and A. Atangana. Reliability analysis incorporating exponentiated inverse Weibull distribution and inverse power law. Quality and Reliability Engineering International, Vol. 37(6), pp. 2399-2422, 2021.

[16]T. N. Sindhu, A. Shafiq and Q.M. Al-Mdallal. Exponentiated transformation of gumbel type-II distribution for modeling COVID-19 data. Alexandria Engineering Journal, Vol. 60 (1), pp. 671–689, 2021.

[17]T. N. Sindhu, A. Shafiq, and Q. M. Al-Mdallal. On the analysis of number of deaths due to Covid- 19 outbreak data using a new class of distributions. Results in Physics, Vol. 21, 03747, 2021.

[18] T. N. Sindhu, A. Shafiq and Z. Hussain. Generalized exponentiated unit Gompertz distribution for modeling arthritic pain relief times data: classical approach to statistical inference. Journal of Biopharmaceutical Statistics,2023

[19] Nassar, M. and Nada, N. The beta generalized Pareto distribution. Journal of Statistics: Advances in Theory and Applications, Vol. 6(1/2), pp. 436–452, 2011.

[20] Lawless, J. F. Statistical models and methods for lifetime data, volume 362. John Wiley & Sons, 2003..